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The number of spanning trees in odd valent circulant graphs☆

Xiebin Chen[a], Qiuying Lin[b], Fuji Zhang[b]

[a] *Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, PR China*
[b] *Department of Mathematics, Xiamen University, Xiamen, Fujian 361005, PR China*

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Abstract

In this paper, we consider the number of spanning trees in circulant graphs. For any class of odd valent circulant graphs $C_{2n}(a_1, a_2, \ldots, a_{k-1}, n)$, where $a_1, a_2, \ldots, a_{k-1}$ are fixed jumps and n varies, some formulas, asymptotic behaviors and linear recurrence relations for the number of its spanning trees are obtained, and some known results on the ones in even valent circulant graphs $C_n(a_1, a_2, \ldots, a_k)$ are improved.

Keywords: Spanning tree; Circulant graph; Linear recurrence relation; Asymptotic behavior

1. Introduction

The number of spanning trees in a graph (network) is an important invariant, it is also an important measure of reliability of a network. The well-known matrix tree theorem (see, e.g. [9]) can be used to count the number of spanning trees for small graphs, but this method is not feasible for large graphs. The number of spanning trees in a graph G is denoted by $t(G)$. For some special classes of graphs, explicit formulas for $t(G)$ have been obtained so far [2,3,5,8,10,12].

The circulant graphs are an important class of graphs, which can be used in the design of local area networks [1]. Let $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant n/2$, where n and $a_i (i = 1, 2, \ldots, k)$ are positive integers. An undirected circulant graph $C_n(a_1, a_2, \ldots, a_k)$ is a regular graph whose set of vertices is $V = \{0, 1, \ldots, n-1\}$ and whose set of edges is

$$E = \{(i, i + a_j \pmod{n}) | i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, k\}.$$

If $a_k < n/2$, then $C_n(a_1, a_2, \ldots, a_k)$ is a $2k$-regular graph; if $a_k = n/2$, then it is a $(2k-1)$-regular one. The formula $t(C_n(1,2)) = nF_n^2, F_n$ being the nth *Fibonacci* number, is well known, its proof can be found in [3,5,7,8,10]. Linear recurrence relations for $t(C_n(1,3))$ and $t(C_n(1,4))$ were given in [10]. For $t(C_n(a_1, a_2, \ldots, a_k))$, where $a_k < n/2$, a formula (see Lemma 3.3 below) and an asymptotic formula were obtained in [5,12], and linear recurrence relations were developed in [12]. The trivalent circulant graph $C_{2n}(1, n)$ is also called the *Möbius* ladder, the formula for $t(C_{2n}(1, n))$ was given in [2, p. 42] (see Example 1 below). In this paper, we consider the number of spanning trees in circulant graphs. For any class of odd valent circulant graphs $C_{2n}(a_1, a_2, \ldots, a_{k-1}, n)$, where $a_1, a_2, \ldots, a_{k-1}$ are fixed jumps and n varies ($2n$ is the number of vertices and n is a non-fixed jump), some formulas, asymptotic behaviors and linear recurrence relations for the number of its spanning trees are obtained, and some known results of [12] on the ones in even valent circulant graphs $C_n(a_1, a_2, \ldots, a_k)$ (where $a_1, a_2, \ldots, a_k$ are fixed jumps) are improved.

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E-mail address: chenxbfz@public.zzptt.fj.cn (X. Chen).

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2. A formula for $t(C_{2n}(a_1, a_2, \ldots, a_{k-1}, n))$

In this section, a formula for $t(C_{2n}(a_1,a_2,\ldots,a_{k-1},n))$ is obtained. To this aim, some lemmas are needed. We start by stating the following two propositions.

Proposition 1 (Biggs [2], Corollary 6.5). *Let G be an h-valent connected graph with n vertices, and let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1} < h$ be the eigenvalues of G. Then*

$$t(G) = \frac{1}{n}\prod_{i=1}^{n-1}(h-\lambda_i).$$

Proposition 2 (Biggs [2], Proposition 3.5). *Suppose $[0, s_2, \ldots, s_n]$ is the first row of the adjacency matrix of a circulant graph G. Let $\varepsilon = \exp(2\pi\sqrt{-1}/n)$. Then the eigenvalues of G are*

$$\lambda_r = \sum_{j=2}^{n} s_j \varepsilon^{r(j-1)}, \qquad r = 0, 1, \ldots, n-1.$$

By Propositions 1 and 2, one can show the following.

Lemma 2.1 (See, e.g., Zhang and Yong [11]). *Let $\varepsilon = \exp(2\pi\sqrt{-1}/n)$. If $(a_1, a_2, \ldots, a_k, n) = 1$, then*

$$t(C_n(a_1,a_2,\ldots,a_k)) = \begin{cases} \dfrac{1}{n}\displaystyle\prod_{j=1}^{n-1}\left(2k - \sum_{i=1}^{k}\left(\varepsilon^{a_i j} + \varepsilon^{-a_i j}\right)\right), & a_k < \dfrac{n}{2}, \\ \dfrac{1}{n}\displaystyle\prod_{j=1}^{n-1}\left(2k - 1 - (-1)^j - \sum_{i=1}^{k-1}\left(\varepsilon^{a_i j} + \varepsilon^{-a_i j}\right)\right), & a_k = \dfrac{n}{2}. \end{cases}$$

Lemma 2.2 (See, e.g., Zhang and Yong [11]). *Let $\varepsilon = \exp(2\pi\sqrt{-1}/m)$. Then*

(1) $\displaystyle\prod_{j=1}^{m-1}(x - \varepsilon^j) = \frac{x^m - 1}{x - 1}, \qquad x \neq 1;$

(2) $\displaystyle\prod_{j=1}^{m-1}(1 - \varepsilon^j) = m.$

Lemma 2.3. *Let $1 \leqslant a_1 < a_2 < \cdots < a_{k-1}$, and let*

$$\varphi(x) = \varphi(x; a_1, a_2, \ldots, a_{k-1}) = \sum_{i=1}^{k-1}(x^{a_{k-1}+a_i} + x^{a_{k-1}-a_i}) - 2kx^{a_{k-1}}.$$

If μ is a root of $\varphi(x)$ with multiplicity h, then so is μ^{-1}, and $|\mu| \neq 1$.

Proof. It is clear that $\varphi(0) = 1$, so $\mu \neq 0$. If $|\mu| = 1$, then

$$\left|\sum_{i=1}^{k-1}\left(\mu^{a_{k-1}+a_i} + \mu^{a_{k-1}-a_i}\right)\right| \leqslant \sum_{i=1}^{k-1} 2 = 2k - 2 < 2k,$$

it follows that $\varphi(\mu) \neq 0$, a contradiction with $\varphi(\mu) = 0$, so $|\mu| \neq 1$. It is clear that if $x \neq 0$, then $\varphi(x) = x^{2a_{k-1}}\varphi(x^{-1})$. Hence $\varphi(\mu^{-1}) = \varphi(\mu) = 0$ and $\mu^{-1} \neq \mu$. Let

$$\varphi(x) = (x - \mu)(x - \mu^{-1})\varphi_1(x).$$

Then

$$\varphi(x^{-1}) = (x^{-1} - \mu)(x^{-1} - \mu^{-1})\varphi_1(x^{-1}) = x^{-2}(x - \mu)(x - \mu^{-1})\varphi_1(x^{-1}),$$

so $\varphi_1(x) = x^{2a_{k-1}-2}\varphi_1(x^{-1})$. If multiplicity $h > 1$, then $\varphi_1(\mu) = 0$ and $\varphi_1(\mu^{-1}) = 0$. Continuing in this fashion, one can show that μ^{-1} is a root with multiplicity h. $\square$

Lemma 2.4. *Let all roots of $\varphi(x)$ in Lemma 2.3 be μ_i and μ_i^{-1}, where $|\mu_i| > 1, i=1,2,\dots,a_{k-1}$, and let $\rho = \exp(\pi\sqrt{-1}/n)$. Then*

$$\prod_{j=1}^{n}\left(2k - 2\sum_{i=1}^{k-1}\cos\frac{a_i(2j-1)\pi}{n}\right)$$

$$=\prod_{j=1}^{n}\left(2k - \sum_{i=1}^{k-1}\left(\rho^{a_i(2j-1)} + \rho^{-a_i(2j-1)}\right)\right)$$

$$=\left(-\prod_{i=1}^{a_{k-1}}(-\mu_i)\right)^{n}\prod_{i=1}^{a_{k-1}}\left(1+\mu_i^{-n}\right)^2.$$

Proof.

$$A := \prod_{j=1}^{2n-1}\left(2k - \sum_{i=1}^{k-1}\left(\rho^{a_i j} + \rho^{-a_i j}\right)\right)$$

$$= (-1)^{2n-1}\prod_{j=1}^{2n-1}\rho^{-a_{k-1}j}\prod_{j=1}^{2n-1}\rho^{a_{k-1}j}\left(\sum_{i=1}^{k-1}\left(\rho^{a_i j} + \rho^{-a_i j}\right) - 2k\right)$$

$$= (-1)^{2n-1}(-1)^{-a_{k-1}(2n-1)}\prod_{j=1}^{2n-1}\varphi(\rho^j)$$

$$= (-1)^{(a_{k-1}-1)(2n-1)}\prod_{j=1}^{2n-1}\prod_{i=1}^{a_{k-1}}(\rho^j - \mu_i)(\rho^j - \mu_i^{-1})$$

$$= (-1)^{(a_{k-1}-1)(2n-1)}\prod_{i=1}^{a_{k-1}}\prod_{j=1}^{2n-1}(\mu_i - \rho^j)(\mu_i^{-1} - \rho^j).$$

By Lemma 2.2,

$$A = (-1)^{(a_{k-1}-1)(2n-1)}\prod_{i=1}^{a_{k-1}}\frac{\mu_i^{2n}-1}{\mu_i - 1}\,\frac{\mu_i^{-2n}-1}{\mu_i^{-1}-1}.$$

Let $\varepsilon = \rho^2$. Similarly, we have

$$B := \prod_{j=1}^{n-1}\left(2k - \sum_{i=1}^{k-1}\left(\varepsilon^{a_i j} + \varepsilon^{-a_i j}\right)\right) = (-1)^{(a_{k-1}-1)(n-1)}\prod_{i=1}^{a_{k-1}}\frac{\mu_i^{n}-1}{\mu_i - 1}\,\frac{\mu_i^{-n}-1}{\mu_i^{-1}-1}.$$

Hence

$$\prod_{j=1}^{n}\left(2k - \sum_{i=1}^{k-1}\left(\rho^{a_i(2j-1)} + \rho^{-a_i(2j-1)}\right)\right) = \frac{A}{B} = \left(-\prod_{i=1}^{a_{k-1}}(-\mu_i)\right)^{n}\prod_{i=1}^{a_{k-1}}\left(1+\mu_i^{-n}\right)^2. \quad \square$$

The following theorem is the main result of this section.

Theorem 2.1. *Let $(a_1, a_2, \dots, a_{k-1}, n) = 1, n > 2a_{k-1}$. If all roots of $\varphi(x)$ in Lemma 2.3 are μ_i and μ_i^{-1}, where $|\mu_i| > 1$, $i = 1,2,\dots,a_{k-1}$, then*

$$t(C_{2n}(a_1,a_2,\dots,a_{k-1},n)) = \frac{1}{2}\,t(C_n(a_1,a_2,\dots,a_{k-1}))\left(-\prod_{i=1}^{a_{k-1}}(-\mu_i)\right)^{n}\prod_{i=1}^{a_{k-1}}\left(1+\mu_i^{-n}\right)^2. \tag{1}$$

Proof. Let $\rho = \exp(\pi\sqrt{-1}/n)$ and $\varepsilon = \rho^2$. By Lemmas 2.1 and 2.4,

$$t(C_{2n}(a_1,a_2,\dots,a_{k-1},n)) = \frac{1}{2n}\prod_{j=1}^{2n-1}\left(2k - 1 - (-1)^j - \sum_{i=1}^{k-1}\left(\rho^{a_i j} + \rho^{-a_i j}\right)\right)$$

$$=\frac{1}{2n}\prod_{j=1}^{n-1}\left(2k-2-\sum_{i=1}^{k-1}\left(\varepsilon^{a_ij}+\varepsilon^{-a_ij}\right)\right)\prod_{j=1}^{n}\left(2k-\sum_{i=1}^{k-1}\left(\rho^{a_i(2j-1)}+\rho^{-a_i(2j-1)}\right)\right)$$

$$=\frac{1}{2}t(C_n(a_1,a_2,\ldots,a_{k-1}))\left(-\prod_{i=1}^{a_k-1}(-\mu_i)\right)^n\prod_{i=1}^{a_k-1}\left(1+\mu_i^{-n}\right)^2. \quad \square$$

3. Linear recurrence relations

In this section, linear recurrence relations for $t(C_n(a_1,a_2,\ldots,a_k))$ and for $t(C_{2n}(a_1,a_2,\ldots,a_{k-1},n))$ are discussed.

Lemma 3.1. *Let $g(x)$ be a polynomial with real coefficients. If all of its roots with modulus greater than* 1 *are x_i, $i=1,2,\ldots,k$, then $\prod_{i=1}^k x_i$ is a real number.*

Proof. Since $g(x)$ is a polynomial with real coefficients, if λ is an imaginary root of $g(x)$ with multiplicity h and modulus greater than 1, then so is its conjugate number $\bar{\lambda}$. By the fact that $\lambda^h\bar{\lambda}^h=|\lambda|^{2h}>1$, it is easy to show the lemma. $\square$

Similar to the proof of Lemma 2.3, one can prove the following.

Lemma 3.2 (Chen [5], Zhang et al. [12]). *Let $k\geqslant 2, 1\leqslant a_1<a_2<\cdots<a_k, (a_1,a_2,\ldots,a_k)=1$, and let*

$$f(x)=f(x;a_1,a_2,\ldots,a_k)=\sum_{i=1}^{k}x^{a_k-a_i}\left(1+x+x^2+\cdots+x^{a_i-1}\right)^2.$$

If λ is a root of $f(x)$ with multiplicity h, then so is λ^{-1}, and $|\lambda|\neq 1$.

Lemma 3.3 (Chen [5], see also Zhang et al. [12, p. 342]). *Let all roots of $f(x)$ in Lemma* 3.2 *be λ_i and λ_i^{-1}, where $|\lambda_i|>1, i=1,2,\ldots,a_k-1$. If $n>2a_k$, then*

$$t(C_n(a_1,a_2,\ldots,a_k))=\frac{n}{f(1)}\left(\prod_{i=1}^{a_k-1}(-\lambda_i)\right)^n\prod_{i=1}^{a_k-1}\left(1-\lambda_i^{-n}\right)^2,$$

where $f(1)=\sum_{i=1}^k a_i^2$.

Lemma 3.4 (See, e.g., Brualdi [4, p. 201]). *Let $x_i\neq 0, i=1,2,\ldots,k$, let*

$$\sigma_1=\sum_{i=1}^{k}x_i,\qquad \sigma_2=\sum_{i<j}x_ix_j,\ \ldots,\qquad \sigma_k=\prod_{i=1}^{k}x_i$$

and let

$$p(x)=\prod_{i=1}^{k}(x-x_i)=x^k-\sigma_1x^{k-1}+\sigma_2x^{k-2}-\cdots+(-1)^k\sigma_k.$$

Assume that

$$s_n=c_1x_1^n+c_2x_2^n+\cdots+c_kx_k^n,\quad n=1,2,\ldots,$$

where $c_1,c_2,\ldots,c_k$ are any constants. Then s_n satisfy the following linear recurrence relation of order k with constant coefficients:

$$s_n-\sigma_1s_{n-1}+\sigma_2s_{n-2}-\cdots+(-1)^k\sigma_ks_{n-k}=0,\ n>k$$

and $p(x)$ is said to be the characteristic polynomial for s_n.

Lemma 3.5 (See, e.g., Feng et al. [6, p. 165]). *Let $\sigma_1,\sigma_2,\ldots,\sigma_k$ be as mentioned in Lemma* 3.4, *and let $\sigma_0=1$. Assume that*

$$s_n=x_1^n+x_2^n+\cdots+x_k^n,\quad n=1,2,\ldots.$$

Then the following Newton's identities hold:

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \cdots + (-1)^{n-1}\sigma_{n-1}s_1 + (-1)^n n\sigma_n = 0,\ \ n = 1,2,\ldots,k.$$

By the induction, one can prove the following.

Lemma 3.6. *Let* $\alpha_i \neq 0, i = 1,2,\ldots,k$, *and let*

$$f_1(x) = \prod_{i=1}^{k}(x-\alpha_i)(x-\alpha_i^{-1}) = \sum_{j=0}^{2k} b_j x^{2k-j},$$

$$f_2(x) = \prod_{i=1}^{k}(x-\alpha_i)(x+\alpha_i^{-1}) = \sum_{j=0}^{2k} c_j x^{2k-j},$$

where b_j (*resp.* c_j), $j = 0,1,\ldots,2k$, *are the coefficients of* $f_1(x)$ (*resp.* $f_2(x)$). *Then*

$$b_j = b_{2k-j}, \quad c_j = (-1)^{k-j}c_{2k-j}, \quad j = 0,1,\ldots,k.$$

For convenience, $f_i(x)$ *is called a polynomial of class* i *for* $i = 1,2$.

Lemma 3.7. *Let* $x_i \neq 0,\ i = 1,2,\ldots,h$, *and let*

$$s_n = \prod_{i=1}^{h}(x_i^n + x_i^{-n}), \quad s_n^- = c\prod_{i=1}^{h}(x_i^n - x_i^{-n}), \quad t_n = \left(\sqrt{-1}\right)^n s_n, \quad t_n^- = \left(\sqrt{-1}\right)^n s_n^-,$$

where $n = 1,2,\ldots$. *Then* s_n, s_n^-, t_n *and* t_n^- *satisfy linear recurrence relations of order* 2^h, *Newton's identities hold for* s_n *and* t_n, *the characteristic polynomial for* s_n *or* s_n^- *is of class* 1, *but the characteristic polynomial for* t_n *or* t_n^- *is of class* 2.

Proof. We write $R \in \Omega$ instead of $(r_1, r_2, \ldots, r_h) \in \{1,-1\}^h$. It is easy to show that

$$s_n = \sum_{R\in\Omega}\left(x_1^{r_1}\cdots x_h^{r_h}\right)^n, \quad s_n^- = \sum_{R\in\Omega} c r_1 r_2 \cdots r_h \left(x_1^{r_1}\cdots x_h^{r_h}\right)^n,$$

$$t_n = \sum_{R\in\Omega}\left(\sqrt{-1}x_1^{r_1}\cdots x_h^{r_h}\right)^n, \quad t_n^- = \sum_{R\in\Omega} c r_1 r_2 \cdots r_h \left(\sqrt{-1}x_1^{r_1}\cdots x_h^{r_h}\right)^n.$$

By Lemmas 3.4 and 3.5, s_n, s_n^-, t_n and t_n^- satisfy linear recurrence relations of order 2^h, and Newton's identities hold for s_n and t_n. The characteristic polynomial for s_n or s_n^- is

$$g_1(x) = \prod_{R\in\Omega}\left(x - x_1^{r_1}\cdots x_h^{r_h}\right),$$

and the characteristic polynomial for t_n or t_n^- is

$$g_2(x) = \prod_{R\in\Omega}\left(x - \sqrt{-1}x_1^{r_1}\cdots x_h^{r_h}\right).$$

It is clear that if $\alpha = x_1^{r_1}\cdots x_h^{r_h}, \beta = \sqrt{-1}x_1^{r_1}\cdots x_h^{r_h}$, then $x_1^{-r_1}\cdots x_h^{-r_h} = \alpha^{-1}$ and $\sqrt{-1}x_1^{-r_1}\cdots x_h^{-r_h} = -\beta^{-1}$. By Lemma 3.6, $g_i(x)$ is of class i for $i = 1,2$. □

The following two theorems are main results in this section.

Theorem 3.1. *Let* $(a_1, a_2, \ldots, a_k) = 1, n > 2a_k$. *If* a_k *is odd, then*

$$t(C_n(a_1,a_2,\ldots,a_k)) = n(s_n^-)^2, \quad \text{where } s_n^- > 0,$$

if a_k *is even, then*

$$t(C_n(a_1,a_2,\ldots,a_k)) = n(t_n^-)^2, \quad \text{where } t_n^- > 0.$$

In addition, s_n^- *and* t_n^- *satisfy linear recurrence relations of order* 2^{a_k-1}, *and the characteristic polynomial for* s_n^- (*resp.* t_n^-) *is of class* 1 (*resp. class* 2).

Proof. Let all roots of $f(x) = f(x; a_1, a_2, \dots, a_k)$ in Lemma 3.2 be λ_i and λ_i^{-1}, where $|\lambda_i| > 1, i = 1, 2, \dots, a_k - 1$. Note that $f(x)$ is a polynomial with real coefficients. Thus, if some λ_i is an imaginary root of $f(x)$ with multiplicity h, then so is its conjugate number $\overline{\lambda_i}$. Since

$$(1 - \lambda_i^{-n})(1 - \overline{\lambda_i}^{-n}) = \left|1 - \lambda_i^{-n}\right|^2 > 0,$$

it is easy to show that $\prod_{i=1}^{a_k-1}(1 - \lambda_i^{-n}) > 0$. By Lemma 3.1, $r := \prod_{i=1}^{a_k-1}(-\lambda_i)$ is a real number. If n is odd, then by Lemma 3.3, it follows that:

$$r = \prod_{i=1}^{a_k-1}(-\lambda_i) > 1. \tag{2}$$

If a_k is odd, then $\sqrt{r} = \prod_{i=1}^{a_k-1}\sqrt{\lambda_i} > 1$ and

$$s_n^- := \frac{1}{\sqrt{f(1)}}\prod_{i=1}^{a_k-1}\left(\sqrt{\lambda_i}\right)^n\prod_{i=1}^{a_k-1}(1 - \lambda_i^{-n}) = \frac{1}{\sqrt{f(1)}}\prod_{i=1}^{a_k-1}\left(\left(\sqrt{\lambda_i}\right)^n - \left(\sqrt{\lambda_i}\right)^{-n}\right) > 0,$$

if a_k is even, then $\sqrt{r} = \sqrt{-1}\prod_{i=1}^{a_k-1}\sqrt{\lambda_i} > 1$ and

$$t_n^- := \frac{1}{\sqrt{f(1)}}\left(\sqrt{-1}\right)^n\prod_{i=1}^{a_k-1}\left(\sqrt{\lambda_i}\right)^n\prod_{i=1}^{a_k-1}(1 - \lambda_i^{-n}) = \left(\sqrt{-1}\right)^n\frac{1}{\sqrt{f(1)}}\prod_{i=1}^{a_k-1}\left(\left(\sqrt{\lambda_i}\right)^n - \left(\sqrt{\lambda_i}\right)^{-n}\right) > 0.$$

By Lemmas 3.3 and 3.7, the theorem follows. □

Remark 3.1. In Lemma 4 and Note of [12] it was proved that $t(C_n(a_1, a_2, \dots, a_k)) = n(h_n)^2$, where h_n satisfy a linear recurrence relation of order 2^{a_k-1}. However, a basic fact that $h_n > 0$ for any n was not proved in [12]. Here we show that $s_n^- > 0$ and $t_n^- > 0$.

Remark 3.2. Consider $t(C_n(1, 5))$. By Theorem 8 of [12], one has to calculate 32 values of h_n and then solve a system of 16 linear equations. By Theorem 3.1, however, one needs only to calculate 24 values of $s_n^-(=h_n)$ and then solve a system of 8 linear equations since the characteristic polynomial for s_n^- is of class 1. Obviously, by Theorem 3.1 we improve the method of [12] for deriving the linear recurrence relation for $t(C_n(a_1, a_2, \dots, a_k))$.

Theorem 3.2. *Let $k \geqslant 3, (a_1, a_2, \dots, a_{k-1}) = 1, n > 2a_{k-1}$. If a_{k-1} is odd, then*

$$t(C_{2n}(a_1, a_2, \dots, a_{k-1}, n)) = \frac{1}{2}t(C_n(a_1, a_2, \dots, a_{k-1}))(s_n)^2 = \frac{n}{2}(s_n^- s_n)^2,$$

if a_{k-1} is even, then

$$t(C_{2n}(a_1, a_2, \dots, a_{k-1}, n)) = \frac{1}{2}t(C_n(a_1, a_2, \dots, a_{k-1}))(t_n)^2 = \frac{n}{2}(t_n^- t_n)^2,$$

where s_n, s_n^-, t_n and t_n^- are positive. In addition, s_n and t_n satisfy linear recurrence relations of order $2^{a_{k-1}}$, s_n^- and t_n^- satisfy linear recurrence relations of order $2^{a_{k-1}-1}$, and Newton's identities hold for s_n and t_n. The characteristic polynomial for s_n or s_n^- is of class 1, but the characteristic polynomial for t_n or t_n^- is of class 2. If a_{k-1} is small, then the initial values of s_n, s_n^-, t_n or t_n^- can be calculated by the following two formulas:

$$\prod_{j=1}^{n}\left(2k - 2\sum_{i=1}^{k-1}\cos\frac{a_i(2j-1)\pi}{n}\right) = \begin{cases}(s_n)^2 & \text{if } a_{k-1} \text{ is odd,}\\(t_n)^2 & \text{if } a_{k-1} \text{ is even,}\end{cases}$$

and

$$\frac{1}{n^2}\prod_{j=1}^{n-1}\left(2k - 2 - 2\sum_{i=1}^{k-1}\cos\frac{2a_i j\pi}{n}\right) = \begin{cases}(s_n^-)^2 & \text{if } a_{k-1} \text{ is odd,}\\(t_n^-)^2 & \text{if } a_{k-1} \text{ is even.}\end{cases}$$

Proof. Consider formula (1) in Theorem 2.1. Similar to the proof of Theorem 3.1, one can show that $\prod_{i=1}^{a_{k-1}}(1+\mu_i^{-n}) > 0$ and

$$s := -\prod_{i=1}^{a_{k-1}}(-\mu_i) > 1. \tag{3}$$

In addition, if a_{k-1} is odd, then

$$s_n := \left(\sqrt{s}\right)^n \prod_{i=1}^{a_{k-1}} (1+\mu_i^{-n}) = \prod_{i=1}^{a_{k-1}} \left(\left(\sqrt{\mu_i}\right)^n + \left(\sqrt{\mu_i}\right)^{-n}\right) > 0,$$

if a_{k-1} is even, then

$$t_n := (\sqrt{s})^n \prod_{i=1}^{a_{k-1}} (1+\mu_i^{-n}) = \left(\sqrt{-1}\right)^n \prod_{i=1}^{a_{k-1}} \left(\left(\sqrt{\mu_i}\right)^n + \left(\sqrt{\mu_i}\right)^{-n}\right) > 0. \tag{4}$$

By Theorems 2.1 and 3.1, Lemmas 2.1, 2.4 and 3.7, the theorem follows. □

4. Examples

In this section, we give three examples.

Example 1. $t(C_{2n}(1,n))$.

The roots of $\varphi(x;1) = x^2 - 4x + 1$ are $2 \pm \sqrt{3}$. It is clear that $t(C_n(1)) = n$. If $n \geqslant 3$, then by Theorem 2.1,

$$t(C_{2n}(1,n)) = \frac{n}{2}\left((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2\right),$$

the formula can be found in [2,3].

Example 2. $t(C_{2n}(1,2,n))$.

The roots of $\varphi(x;1,2) = x^4 + x^3 - 6x^2 + x + 1$ are

$$\frac{1}{4}\left(-1 - \sqrt{33} \pm \sqrt{18 + 2\sqrt{33}}\right)$$

and

$$\frac{1}{4}\left(-1 + \sqrt{33} \pm \sqrt{18 - 2\sqrt{33}}\right).$$

By Theorem 3.2 and formula (4),

$$t(C_{2n}(1,2,n)) = \tfrac{1}{2} t(C_n(1,2))(t_n)^2 = \frac{n}{2}(F_n)^2 (t_n)^2,$$

where F_n is the nth *Fibonacci* number and

$$(t_n)^2 = \left(\left(\frac{\sqrt{33}+1+\sqrt{18+2\sqrt{33}}}{4}\right)^n + \left(\frac{\sqrt{33}+1-\sqrt{18+2\sqrt{33}}}{4}\right)^n + 2(-1)^n\right)$$
$$\times \left(\left(\frac{\sqrt{33}-1+\sqrt{18-2\sqrt{33}}}{4}\right)^n + \left(\frac{\sqrt{33}-1-\sqrt{18-2\sqrt{33}}}{4}\right)^n + 2\right).$$

In addition, t_n satisfy a linear recurrence relation of order 4

$$t_n - \sigma_1 t_{n-1} + \sigma_2 t_{n-2} + \sigma_1 t_{n-3} + t_{n-4} = 0, \quad n > 4.$$

It is clear that $t_1 = \sqrt{6}$, $t_2 = 8$. By Newton's identities, one can step by step derive the following:

$$\sigma_1 = \sqrt{6}, \quad \sigma_2 = -1, \quad t_3 = 6\sqrt{6}, \quad t_4 = 34$$

hence

$$t_n = \sqrt{6}(t_{n-1} - t_{n-3}) + t_{n-2} - t_{n-4}, \quad n > 4.$$

Example 3. $t(C_{2n}(1,3,n))$.

By Theorem 3.2,

$$t(C_{2n}(1,3,n)) = \frac{n}{2}(s_n^- s_n)^2,$$

where s_n^- (resp. s_n) satisfy a linear recurrence relation of order 4 (resp. 8). Assume that

$$s_n^- - a s_{n-1}^- - b s_{n-2}^- - a s_{n-3}^- + s_{n-4}^- = 0, \quad n > 4.$$

By Theorem 3.2,

$$(s_n^-)^2 = \frac{1}{n^2}\prod_{j=1}^{n-1}\left(4 - 2\cos\frac{2j\pi}{n} - 2\cos\frac{6j\pi}{n}\right).$$

It can be verified that

$$s_2^- = \sqrt{2}, \quad s_3^- = 1, \quad s_4^- = 2\sqrt{2}, \quad s_5^- = 5, \quad s_6^- = 5\sqrt{2}, \quad s_7^- = 13$$

and that the solution to the following system of equations

$$(5+1)a + 2\sqrt{2}b - \sqrt{2} = 5\sqrt{2}, \quad (5\sqrt{2} + 2\sqrt{2})a + 5b - 1 = 13$$

is $a = \sqrt{2}, b = 0$. Hence

$$s_n^- = \sqrt{2}(s_{n-1}^- + s_{n-3}^-) - s_{n-4}^-, \qquad n > 4.$$

Assume that

$$s_n - \sigma_1 s_{n-1} + \sigma_2 s_{n-2} - \sigma_3 s_{n-3} + \sigma_4 s_{n-4} - \sigma_3 s_{n-5} + \sigma_2 s_{n-6} - \sigma_1 s_{n-7} + s_{n-8} = 0, \qquad n > 8.$$

By Theorem 3.2,

$$(s_n)^2 = \prod_{j=1}^{n}\left(6 - 2\cos\frac{(2j-1)\pi}{n} - 2\cos\frac{(6j-3)\pi}{n}\right),$$

it can be verified that $s_1 = \sqrt{10}$, $s_2 = 6$, $s_3 = 7\sqrt{10}$, $s_4 = 36$. By Newton's identities, one can step by step derive the following: $\sigma_1 = \sqrt{10}$, $\sigma_2 = 2$, $\sigma_3 = \sqrt{10}$, $\sigma_4 = 8$, $s_5 = 25\sqrt{10}$, $s_6 = 198$, $s_7 = 139\sqrt{10}$, $s_8 = 1016$. Hence

$$s_n = \sqrt{10}(s_{n-1} + s_{n-3} + s_{n-5} + s_{n-7}) - (2s_{n-2} + 8s_{n-4} + 2s_{n-6} + s_{n-8}), \qquad n > 8$$

and the initial values s_i, $i = 1, 2, \ldots, 8$, are given as above.

5. Asymptotic behaviors

In this section, asymptotic behaviors for $t(C_{2n}(a_1, a_2, \ldots, a_{k-1}, n))$ are developed.

Theorem 5.1. *Let* $k \geqslant 3, (a_1, a_2, \ldots, a_{k-1}, n) = 1$. *If all roots of* $\varphi(x; a_1, a_2, \ldots, a_{k-1})$ *with modulus greater than* 1 *are* $\mu_i, i = 1, 2, \ldots, a_{k-1}$, *then*

$$t(C_{2n}(a_1, a_2, \ldots, a_{k-1}, n)) \sim \tfrac{1}{2}\, t(C_n(a_1, a_2, \ldots, a_{k-1}))s^n, \qquad n \to \infty,$$

where $s = -\prod_{i=1}^{a_{k-1}}(-\mu_i) > 1$. *Moreover, if* $(a_1, a_2, \ldots, a_{k-1}) = 1$, *and all roots of* $f(x; a_1, a_2, \ldots, a_{k-1})$ *with modulus greater than* 1 *are* $\lambda_i, i = 1, 2, \ldots, a_{k-1} - 1$, *then*

$$t(C_{2n}(a_1, a_2, \ldots, a_{k-1}, n)) \sim \frac{n}{2f(1)}(rs)^n, \qquad n \to \infty,$$

where

$$r = \prod_{i=1}^{a_{k-1}-1}(-\lambda_i) > 1, \qquad f(1) = \sum_{i=1}^{k-1} a_i^2.$$

Proof. It is clear that if $n \to \infty$, then

$$\prod_{i=1}^{a_{k-1}}\left(1 + \mu_i^{-n}\right) \to 1 \quad \text{and} \quad \prod_{i=1}^{a_{k-1}-1}\left(1 - \lambda_i^{-n}\right) \to 1.$$

By formulas (1)–(3) and Lemma 3.3, one can easily prove the theorem. $\square$

Remark 5.1. Let h_n be as mentioned in Remark 3.1. In Section 4 of [12], it was shown that $h_n \sim c\phi^n$, where $\alpha = 1/\phi$ is the root of the polynomial $Q(x)$ of degree 2^{a_k-1} with $|\alpha|$ being unique minimum modulus, and c is a number depending on α. By Lemma 3.3, it is easy to show that c and ϕ satisfy the following equations:

$$c = 1/\sqrt{f(1)} = 1 \Bigg/ \sqrt{\sum_{i=1}^{k} a_i^2}, \qquad \phi = \sqrt{r}, \qquad \text{where } r = \prod_{i=1}^{a_k-1}(-\lambda_i) > 1.$$

Note that $Q(x)$ is of degree 2^{a_k-1}, but $f(x)$ in Lemma 3.2 is of degree $2a_k - 2$. Hence the method of [12] for determining c and ϕ is more difficult than ours.

To calculate or estimate $t(C_n(a_1, a_2, \ldots, a_k))$, where $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant n/2$, the most important step is to find the roots of $\varphi(x) = \varphi(x; a_1, a_2, \ldots, a_{k-1})$ in Lemma 2.3 and $f(x) = f(x; a_1, a_2, \ldots, a_k)$ in Lemma 3.2. Here we provide an approach to reduce the order of equations $\varphi(x) = 0$ and $f(x) = 0$. In fact we introduce new equations of half the order whose roots will provide the roots of the original equations.

We have the following.

Theorem 5.2. *Let* $g(i) = x^i + x^{-i}$, $y = g(1) = x + x^{-1}$. *Then*

$$(1)\ g(2k+1) = y^{2k+1} - (2k+1)y^{2k-1} + \sum_{i=0}^{k-2}(-1)^i\left(\binom{2k-i}{i+2} - \binom{2k-2-i}{i}\right)y^{2k-3-2i},$$

where $k \geqslant 0$;

$$(2)\ g(2k) = y^{2k} - (2k)y^{2k-2} + \sum_{i=1}^{k-2}(-1)^{i+1}\left(\binom{2k-i}{i+1} - \binom{2k-2-i}{i-1}\right)y^{2k-2-2i} + (-1)^k 2,$$

where $k \geqslant 1$.

In the above two equalities, all exponents of y *must be non-negative, otherwise the coefficients of* y^i *are considered zeros.*

Proof. It is clear that

$$g(i+4) = g(i+2)g(2) - g(i) = g(i+2)(y^2 - 2) - g(i).$$

By induction on i (there are two cases: $i = 2k$ and $2k+1$), one can show the theorem. □

The reduced procedure can be illustrated by the following.

Example 4. An asymptotic formula for $t(C_{2n}(1, 3, n))$.

By Theorem 5.2, we have

$$\varphi(x; 1, 3) = x^6 + x^4 - 6x^3 + x^2 + 1 = x^3(x^3 + x - 6 + x^{-1} + x^{-3})$$
$$= x^3(g(3) + g(1) - 6) = x^3(y^3 - 2y - 6).$$

Solving the equation $y^3 - 2y - 6 = 0$, we have

$$y_1 \doteq 2.179981072, \ y_{2,3} \doteq -1.089990536 \pm 1.250695049\sqrt{-1}.$$

Then solving the following three equations:

$$x + x^{-1} = y_i, \quad i = 1, 2, 3,$$

we have three roots of $\varphi(x; 1, 3) = 0$ with modulus greater than 1 as follows:

$$\mu_1 \doteq 1.523671719, \ \mu_{2,3} \doteq -0.857216992 \pm 1.716916161\sqrt{-1},$$

so $s = \mu_1\mu_2\mu_3 \doteq 5.611107108$. Similarly, the product of two roots of $f(x; 1, 3) = 0$ with modulus greater than 1 is $r \doteq 2.890053639$. By Theorem 5.1, we have

$$t(C_{2n}(1, 3, n)) \sim \frac{n}{20}(rs)^n \doteq \frac{n}{20} \times 16.216400519^n, \quad n \to \infty.$$

Theorem 5.3. *If* $(a_1, a_2, \ldots, a_{k-1}) = 1, (d, n) = 1,$ *then*

$$t(C_{2n}(a_1 d, a_2 d, \ldots, a_{k-1} d, n)) \sim t(C_{2n}(a_1, a_2, \ldots, a_{k-1}, n)), \quad n \to \infty.$$

Table 1
Some values of r and s

$\{a_1,a_2\}$	r	s	$\{a_1,a_2\}$	r	s
$\{1,2\}$	2.61803399	5.55193337	$\{3,8\}$	3.16417314	5.62550850
$\{1,3\}$	2.89005364	5.61110711	$\{3,10\}$	3.17918650	5.62551066
$\{1,4\}$	3.01652556	5.62258730	$\{4,5\}$	3.12952227	5.62546317
$\{1,5\}$	3.08213728	5.62489905	$\{4,7\}$	3.15874149	5.62550685
$\{1,6\}$	3.11971341	5.62537940	$\{4,9\}$	3.17545950	5.62551051
$\{1,7\}$	3.14301074	5.62548194	$\{5,6\}$	3.15554267	5.62550562
$\{1,8\}$	3.15838129	5.62550434	$\{5,7\}$	3.16496318	5.62550922
$\{1,9\}$	3.16903035	5.62550933	$\{5,8\}$	3.17245060	5.62551034
$\{1,10\}$	3.17670224	5.62551047	$\{5,9\}$	3.17840124	5.62551067
$\{2,3\}$	2.96557263	5.62028983	$\{6,7\}$	3.17074974	5.62551021
$\{2,5\}$	3.09610667	5.62520180	$\{7,8\}$	3.18038401	5.62551074
$\{2,7\}$	3.14695418	5.62549313	$\{7,9\}$	3.18421556	5.62551078
$\{2,9\}$	3.17051305	5.62550979	$\{7,10\}$	3.18746930	5.62551080
$\{3,4\}$	3.07959562	5.62504083	$\{8,9\}$	3.18686371	5.62551080
$\{3,5\}$	3.11308583	5.62538186	$\{9,10\}$	3.19142702	5.62551081
$\{3,7\}$	3.15250537	5.62550186			

Proof. Suppose that all roots of $\varphi(x;a_1,a_2,\ldots,a_{k-1})$ with modulus greater than 1 are μ_i, $i=1,2,\ldots,a_{k-1}$, and all roots of $\varphi(x;a_1d,a_2d,\ldots,a_{k-1}d)$ with modulus greater than 1 are α_i, $i=1,2,\ldots,a_{k-1}d$. Since

$$\varphi(x;a_1,a_2,\ldots,a_{k-1})=\prod_{i=1}^{a_{k-1}}(x-\mu_i)(x-\mu_i^{-1}),$$

it follows that

$$\varphi(x;a_1d,a_2d,\ldots,a_{k-1}d)=\varphi(x^d;a_1,a_2,\ldots,a_{k-1})=\prod_{i=1}^{a_{k-1}}(x^d-\mu_i)(x^d-\mu_i^{-1}).$$

Suppose all roots of $x^d-\mu_i$ are $\beta_{i,j}$, $j=1,2,\ldots,d$, $i=1,2,\ldots,a_{k-1}$. Then

$$\begin{aligned}\prod_{i=1}^{a_{k-1}d}(-\alpha_i)&=(-1)^{a_{k-1}d}\prod_{i=1}^{a_{k-1}d}\alpha_i=(-1)^{a_{k-1}d}\prod_{i=1}^{a_{k-1}}\prod_{j=1}^{d}\beta_{i,j}\\&=(-1)^{a_{k-1}d}\prod_{i=1}^{a_{k-1}}(-1)^{d+1}\mu_i=\prod_{i=1}^{a_{k-1}}(-\mu_i).\end{aligned}$$

Note that (see, e.g., Note of [12]) $C_n(a_1d,a_2d,\ldots,a_{k-1}d)$ is isomorphic to $C_n(a_1,a_2,\ldots,a_{k-1})$. By Theorem 5.1,

$$\begin{aligned}&t(C_{2n}(a_1d,a_2d,\ldots,a_{k-1}d,n))\\&\quad\sim\frac{1}{2}\,t(C_n(a_1d,a_2d,\ldots,a_{k-1}d))\left(-\prod_{i=1}^{a_{k-1}d}(-\alpha_i)\right)^n\\&\quad=\frac{1}{2}\,t(C_n(a_1,a_2,\ldots,a_{k-1}))\left(-\prod_{i=1}^{a_{k-1}}(-\mu_i)\right)^n\\&\quad\sim t(C_{2n}(a_1,a_2,\ldots,a_{k-1},n)),\qquad n\to\infty.\qquad\square\end{aligned}$$

It is possible that

$$t(C_{2n}(a_1d,a_2d,\ldots,a_{k-1}d,n))\neq t(C_{2n}(a_1,a_2,\ldots,a_{k-1},n)).$$

For example, if n is odd, then by Theorem 2.1,

$$t(C_{2n}(2,n))=\frac{n}{2}\left((2+\sqrt{3})^n+(2-\sqrt{3})^n-2\right)<t(C_{2n}(1,n)).$$

Let r and s be as mentioned in Theorem 5.1. For $1 \leqslant a_1 < a_2 \leqslant 10$ with $(a_1, a_2) = 1$, we used Mathematica to calculate r and s, the results are displayed in Table 1.

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